Let $y_n = \frac{x_n}{n\|x_n\|}$ for each n.

Then $y_n \to 0$ as $n \to \infty$.

$\Rightarrow \|y_n\| \to 0$ as $n \to \infty$

Now →

$$\|T(y_n)\| = \left\| T\left(\frac{x_n}{n\|x_n\|}\right)\right\|$$

$$= \left\| \frac{1}{n\|x_n\|} T(x_n)\right\|$$

, using $T(\alpha x) = \alpha T(x)$

$$= \frac{1}{n\|x_n\|} \|T(x_n)\| \quad \to (3)$$

Since from (2),

$$\|T(x_n)\| > n\|x_n\|$$

∴ From (3), we have →

$$\|T(y_n)\| > 1$$

$\Rightarrow T(y_n)$ does not tend to 0 as $n \to \infty$

∴ $\langle y_n \rangle$ converges to 0

but $\langle T(y_n)\rangle$ does not converge to 0

$\Rightarrow$ T is not cont. at 0

which is a contradiction

∴ T must be bdd.

(Proved)

Def^n Let M be a closed linear sub-space of a NLS N. Let ϕ be a mapping from N onto N/M, defined by→

$$\phi : N \xrightarrow{\text{onto}} N/M$$

$$\phi(x) = x + M \quad \forall x \in N$$

Then ϕ is called natural mapping or homomorphism of N onto N/M.

Th^m:- Natural mapping 'ϕ' as defined above is a bdd. (or continuous) linear trans. with $\|\phi\| \leq 1$

Proof:- Since M is closed, ∴ N/M is a NLS. Define the norm of a coset $x + M \in N/M$ as —

$$\|x+M\| = \inf\{\|x+m\| \mid m \in M\}$$

First we shall show that ϕ is linear.

Let $x, y \in N$ and α, β any scalars

$$\phi(\alpha x + \beta y) = (\alpha x + \beta y) + M, \quad \text{by def}^n \text{ of } '\phi'$$

$$= (\alpha x + M) + (\beta y + M), \quad \because M + M = M$$

$$= \alpha(x + M) + \beta(y + M), \quad \because \alpha M = M \ \& \ \beta M = M$$

$$= \alpha\phi(x) + \beta\phi(y)$$

∴ ϕ is linear.

ϕ is continuous

$\|\phi(x)\| = \|x+M\| = \inf\{\|x+m\| \mid m \in M\}$

For $m=0$, we have, $\leq \|x+m\|, \forall m \in M$

$\|\phi(x)\| \leq \|x\| = 1 . \|x\|, \forall x \in N$ $\longrightarrow (1)$

$\Rightarrow \phi$ is bdd.

$\Rightarrow \phi$ is continuous.

Now to show $\|\phi\| \leq 1$.

$\|\phi\| = \sup\{\|\phi(x)\| \mid x \in N, \|x\| \leq 1\}$

$\leq \sup\{\|x\| \mid x \in N, \|x\| \leq 1\}$, from (1)

≤ 1

i.e. $\|\phi\| \leq 1$

(Proved)

Defn Let $N, N' \to$ NLS

$T: N \to N'$ L.T.

Then Null space (or Kernel) of T is defined as -

$Ker(T) = \{x \mid x \in N, T(x) = 0\}$

i.e. set of those elements of N which are mapped to '0' of N'.

Ex- Show that Ker(T) is a linear manifold (subspace). Also show that Ker(T) is closed if T is continuous

Soln. Let $x, y \in \ker(T) \Rightarrow Tx=0, Ty=0$
$\alpha, \beta \to$ any scalars.

Then
$$T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$$
$$=\alpha.0+\beta.0=0$$

$\therefore$ $\alpha x+\beta y \in \ker(T)$

$\therefore$ $\ker(T)$ is linear manifold

Let T be cont. Then

$\ker(T)=T^{-1}\{0\}$, as $T(x)=0 \Rightarrow x=T^{-1}\{0\}$

Since $\{0\}$ is closed in N' and T is continuous

$\therefore$ $\ker(T)$ is also closed

(Proved).

Exm. Let N, N' be NLS and let T be L.T. from N onto N'. Show that the following are equivalent —

(i) T is bdd (ii) T is uniformly cont. on N (iii) T is cont. at some point of N.

Soln. (i) $\Rightarrow$ (ii) Given T is bdd.

$\therefore$ $\exists$ $M>0$ s.t. $\|T(x)\| \leq M\|x\|$ $\forall x \in N$ $\to$ (1)

To show that T is unif. cont.

Let $\epsilon>0$ be given. Choose $\delta=\frac{\epsilon}{M}$

Then for $x_1, x_2 \in N$ and $\|x_1 - x_2\| < \delta$, we have →

$$\|T(x_1) - T(x_2)\| = \|T(x_1 - x_2)\|$$
$$\leq M \|x_1 - x_2\|$$
$$< M \frac{\epsilon}{M} = \epsilon$$

$\Rightarrow T$ is unif. cont.

(ii) ⇒ (iii) Since T is unif. cont.

∴ it is continuous at each & every point of N ⇒ cont. at some point of N

(iii) ⇒ (i) Let T be cont. at some point $x_0 \in N$. Then for $\epsilon = 1$, $\exists\ \delta > 0$ s.t.

$$\|x - x_0\| < \delta \Rightarrow \|T(x) - T(x_0)\| < 1 \quad \rightarrow (1)$$

For any $y \in N$, let $z = \frac{hy}{\|y\|}$ with $0 < h < \delta$

Then →

$$\|(z + x_0) - x_0\| = \|z\|$$
$$= \left\| \frac{hy}{\|y\|} \right\|$$
$$= \frac{1}{\|y\|} h \|y\|$$
$$= h < \delta$$

∴ From (1), we get →

$$\|T(z + x_0) - T(x_0)\| < 1 \quad \rightarrow (2)$$

Now →

$$\|T(z + x_0) - T(x_0)\| = \|T(z + x_0 - x_0)\|$$
$$= \|T(z)\|$$

$$= \left\| T\left(\frac{h y}{\|y\|}\right)\right\|$$

$$= \left\| \frac{h}{\|y\|} T(y)\right\|$$

$$= \frac{h}{\|y\|} \|T(y)\| \quad \text{---}(3)$$

From (2) & (3), we get —

$$\frac{h}{\|y\|} \|T(y)\| < 1$$

i.e. $\|T(y)\| < h^{-1}\|y\|$

$\Rightarrow$ T is bdd.

(Proved)